B"H

Torah and Mathematics

When Two Triangles Make a Square

from Harav Yitzchak Ginsburgh

Hero and Heroine

The history of the Hasmonean revolt against the Seleucid Greeks tells of a hero and heroine. The hero is *Matityahu*, the High Priest of the time, who with his five sons led the revolution. The heroine is *Yehudit*, who seduced, sedated, and killed the Greek general Alaforni.

Aside from their bravery against the Greeks, these two heroes have a mathematical trait in common. The *gematria* of *Matityahu* (מַתִּתְיָהוּ) is 861 and the *gematria* of *Yehudit* (יְהוּדִית) is 435. Both 861 and 435 are triangular numbers:

$$435 = \triangle 29$$

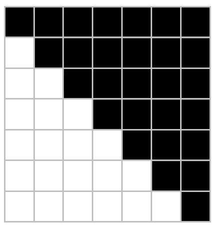
$$861 = \triangle 41$$

Triangular Numbers and Square Numbers

Triangular numbers share a strong relationship with square numbers. One of the most basic definitions of a square number is that it is the sum of two consecutive triangular numbers. More rigorously:

$$n^2 = \triangle n + \triangle (n-1)$$

And pictured geometrically,



(The black squares are $\triangle 7$ and the white squares are $\triangle 6$, and their sum is 7^2)

One question that arises in the context of this discussion of the relationship between triangular and square numbers is, Are there are other pairs of triangular numbers that together equal a square number? As it turns out, there are many such families of triangular numbers.

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Indeed, our own pair of triangular numbers, 435 and 861, together equal 1296, which is the square of 36 (the number of candles we light on Chanukah, from 1 on the first night to 8 on the last night), or,

$$\triangle 29 \perp \triangle 41 = 36^2$$

We would like to see if this pair is part of a larger family of triangular numbers whose sum is a square number.

Generalization

In order to find the family of triangular pairs, the first thing to note is that 36, the square root of the sum of our two triangular numbers is itself a square number (36 is also a triangular number, the triangle of 8 as noted above with regard to the 36 candles that we light on Chanukah, but this understanding of the significance of 36 in the above equation will not produce a general rule). We can then write that:

$$\triangle 29 \perp \triangle 41 = (6^2)^2$$

So, the first property we surmise is that the sum of our triangle pairs will be not just any square number, but a square number whose root is a square.

Let us look at $(5^2)^2 = 625$. It is easy to find that there indeed exist two triangular numbers (other than the triangles of 25 and 24) whose sum is equal to 625:

$$\triangle 19 \perp \triangle 29 = 625 = (5^2)^2$$

Observing the similarity of proportions between these numbers and the previous ones a general picture already seems to emerge. Working backwards we can construct the following table of similar values that satisfy the equation, $\triangle a \perp \triangle b = c^2$, where $c = n^2$:

a	b	C
29	41	36
19	29	25
11	19	16
5	11	9
1	5	4
-1	1	1
-1	-1	0

Again, note that the c column is simply the square numbers.

Let us highlight a few properties of this table:

- The differences between the values in the a column are the even integers.

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- The differences between the values in the b column are the even integers again, starting a step back.
- The differences between the values in the c column are the odd integers, a well-known fact that the differences between the square numbers are the odd integers.
- Every value of b becomes the value of a in the row above.
- The difference between each b and its corresponding a is always twice the square root of c (in our case in particular, the difference between 41 and 29 is 12, 2 times 6, the root of 36). The lower difference, between c and a, is always 2 more than the upper difference between b and c.
- Every four consecutive numbers in either the a or b column equals a square number (the squares of the even integers).

From this table we can extrapolate the general equation describing our family of triangular number pairs whose sum is a square number:

For any integer n: $\triangle(n^2 - n - 1) + \triangle(n^2 + n - 1) = n^4$

Another relationship that can be induced from this table can be expressed mathematically in the following form:

$$\sum_{n=1}^{k} \Delta b_n - \Delta a_n = \Delta b_n$$

Masculine and Feminine

One of the important topics in Torah is that of pairing different entities, concepts, etc., into male-female pairs. This extends to number theory, where according to Torah integers are either masculine or feminine and therefore can be paired. One of the most ubiquitous such pairs found in Torah is 7 and 13, where 13 is the relatively masculine number and 7 the relatively feminine.

Obviously, *Matityahu* and *Yehudit*, the hero and heroine of Chanukah constitute a male-female pair. And this is reflected in the above generalization of their numerical values.

If we sum up the values of the five numbers in the a column from 1 to 29 (recall that $\triangle 29 = 435$, or *Yehudit*), so as to include only positive integers, we get: $1 + 5 + 11 + 19 + 29 = 65$. But, 65 is a multiple of 13 (the average value of the five numbers), making it relatively masculine.

If we sum up the values of the five corresponding numbers in the b column from 5 to 41 (recall that $\triangle 41 = 861$, or *Matityahu*), we get: $5 + 11 + 19 + 29 + 41 = 105$. (Note that all of the numbers until 41 are prime whereas the next number, 55 – for a = 41 b = 55 – is composite; 55 is the product of 5 and 11.) But, 105 is a multiple of 7 (21, the "golden pair" of 13, is the average value of these five numbers), making it relatively feminine.

So, we have found that the a column, representing the feminine—*Yehudit*—reflects a masculine quality (being a multiple of 13), *Yehudit* indeed acted as a male in killing the Greek general Alaforni. On the other hand the b column, representing the masculine—*Matityahu*—reflects a feminine quality (being a multiple of 7), *Matityahu* inspired his five sons to wage war against the mighty army of the Greeks, just as a woman, with her strong faith in God, inspires men to fight the wars of God. This is a well-known phenomena called “switching places” (אחליפו דוכתייהו) in the *Zohar* and signifies a state of rectification and maturity, where the masculine and feminine can reverse roles in certain contexts.

More about 29 and 41

Of all of the pairs of integers a and b enumerated above the pair 29 and 41 (for n = 6), the Chanukah pair, possesses a unique property: 29^2, 841, is the mid-point of 41^2, 1681. This means that 841 is not only a square number, the square of 29, but also an "inspirational number," the sum of two consecutive squares: $29^2 = 841 = 441 + 400 = 21^2 + 20^2$, where 21 + 20 = 41, the pair of 29.

Now, contemplating the above table we saw the rule that in every row c – a is 2 more than b – c. In the case of a = 29, b = 41, and c = 36, c – a = 7 and b – c = 5. Since 7 is the value of the letter zayin (ז) and 5 is the value of the letter hei (ה), when we add these two numbers together we form the word "this [is]" (**זה**). This word represents the unique level of prophecy given to Moshe Rabbeinu. Altogether, Moshe starts his prophecy with this word 8 times, corresponding to the 8 days of Chanukah.

But, notice that these two numbers, 7 and 5, possess the same special property just noted regarding 29 and 41, namely, that 5^2, 25, is the mid-point of 7^2, 49! As stated above, this implies that 25 in addition to being a square number ($25 = 5^2$), is also an inspirational number, the sum of the two consecutive squares, the squares of 3 and 4 (where 3 + 4 = 7, the pair of 5).

We might wonder if there is any other pair of numbers between 5 and 7 and 29 and 41 that possesses this property. It turns out that these numbers are unique. The next pair exhibiting this property is 169 and 239. In order to rigorously understand the pattern involved, one has to be familiar with Pell's theorem. But, non-rigorously it can demonstrated to be a recursive function with each new pair being 6 times the previous pair minus the pair preceding the previous one. In the case of 169 and 239:

$169 = 6 \cdot 29 - 5$

$239 = 6 \cdot 41 - 7$

Chanukah celebrates the victory of the Torah over Hellenistic philosophy and culture. The victory brings to light (in the candles of Chanukah) that all the sparks of true wisdom to be found in Greek culture are rooted in the Torah and reflected in the souls of Israel. Here we have discovered how the two most basic Pythagorean triplets x, y, z (3, 4, 5 and 20, 21, 29, where x and y are consecutive numbers) relate to the names of the heroes of Chanukah.

Triangular-Square Numbers

We noted above that 36 (the number of candles that we light during the 8 days of Chanukah) is a triangular-square number, meaning it both a triangle ($\triangle 8$) and a square (6^2). The next triangular-square after 36 is 1225, which is both $\triangle 49$ ($49 = 7^2$) and 35^2 (note that $35 = 5 \cdot 7$).

1225 also has special significance in Judaism as it is the sum of days that we count during the Counting of the Omer. Though the Counting of the Omer lasts for 49 days (from the end of the first day of Passover to the day before Shavu'ot) the way that we count explicitly states that we are adding days. We do not say "Today is the first day of the Omer," "Today is the second day of the Omer," etc. but rather on the first day we say, "Today is one day to the Omer"; on the second day we say, "Today are two days to the Omer" etc., and we do not "today is the second day of the Omer" and so on.

Now, let us fill in the table for c = 1225. It is easy to calculate that for c = 1225, a will be 1189 (36 less than 1225). But, $1189 = 29 \cdot 41$!

Chanukah and Tzadikim

Now, let us imagine that the 8 days of Chanukah correspond to the 8 rows of our table from n = 1 to n = 8. We noted above that the sum of any 4 consecutive numbers in either the a or b columns will always equal a square number. For the b column (all positive numbers from 1) this means $1 + 5 + 11 + 19 = 36 = 6^2$ (once more the number of candles that we light on Chanukah) and $29 + 41 + 55 + 71 = 196 = 14^2$. Together, all 8 numbers equal 232, the numerical value of God's first two words in creation: "Let there be light" (יְהִי אוֹר), the light of Chanukah! The sum of the first 8 squares (the c column) = 204, whose square, 41616, is the next triangular-square number after 1225; $41616 = 204^2 = \triangle 288$(!)

204 is the numerical value of the word *tzadik* (צַדִּיק), of whom there are 36 in every generation. Thus, one of the secrets inherent in the 36 candles of Chanukah is that each of the 36 *tzadikim* of the generation shines his unique light to all the souls of the

generation, tells his unique story, and sings his unique song, through one of the candles of Chanukah!